QN → State and Prove Cauchy's general Principle of Convergence.

Ans → Statement :– The necessary and sufficient condition for a real sequence to be convergent is that it is a Cauchy's sequence.

Proof :–

For necessary condition

i.e. every convergent sequence is Cauchy ~~Convergent~~ sequence.

Let, us suppose that sequence $\{a_n\}$ converges to $l$. Then given $\epsilon > 0$, there exists a natural no. $p$, such that,

$$|a_n - l| < \frac{\epsilon}{2} \text{ for all } n \geq p$$

Consequently $m, n \geq p$ implies

$$|a_n - a_m| = |(a_n - l) + (l - a_m)|$$
$$\leq |a_n - l| + |l - a_m|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, $\{a_n\}$ is a Cauchy's sequence.

i.e. every convergent sequence

For Sufficient condition :-

i.e. every cauchy's sequence of real number is a convergent sequence.

To Prove this let us suppose that $\{a_n\}$ is a Cauchy sequence of real no. Then given $\epsilon > 0$, there exists a natural no. $p$ such that,

$$|a_n - a_m| < \frac{\epsilon}{2} \text{ for all } n, m \geq p \quad \text{— (a)}$$

On taking $m = p$ in eqn. (a), we have $n \geq p$.

$$|a_n - a_p| < \frac{\epsilon}{2}$$

$$a_p - \frac{\epsilon}{2} < a_n < a_p + \frac{\epsilon}{2} \text{ for } n \geq p \quad \text{—— (b)}$$

Now, let us Consider a set

$$S = \{z \in R : z < a_n \text{ for infinity } n\}.$$

from above we see that,

$a_p - \frac{\epsilon}{2}$ is in $S$. Hence, $S$ is non empty set.

It is also clear that,

$a_p + \frac{\epsilon}{2} >$ each element of $S$. Hence, $S$ is bounded above.

Now, we know from axioms of least bound, l.u.b. $S$ exist as a real no.

Let, $l =$ l.u.b $S$

$$\therefore l \in \left[a_p - \frac{\epsilon}{2}, a_p + \frac{\epsilon}{2}\right]$$

Hence, $|a_p - l| \leq \epsilon/2$ —— (c)

Now, for all $n \geq p$, we have

$$|a_n - l| = |a_n - a_p + a_p - l|$$

$$< |a_n - a_p| + |a_p - l|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

from (b) & (c)

Hence, $\{a_n\}$ Converges to $l$.

Thus every Cauchy sequence of real no. is a Convergent sequence.

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Q.No- State and Prove Taylor's theom with Cauchy's form of remainder.

Ans. Statement :- If $f$ be a real valued function on $[a, a+h]$ such that

(i) all the derivatives upto $(n-1)$ are Continuous in closed interval $[a, a+h]$

(ii) $f^{n}(x)$ exists in open interval $]a, a+h[$

then,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1}f^{(n-1)}(a) + \frac{h^n}{\lfloor n-1}(1-\theta)^{n-1} f^{(n)}(a+\theta h),$$

where, $0<\theta<1$

Proof :- To prove this theom, let us Consider a function "F" defind by

$$F(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{\lfloor 2}f''(x) + \cdots + \frac{(a+h-x)^{n-1}}{\lfloor n-1}f^{(n-1)}(x) + (a+h-x)A.$$

Where, A is Const= to be determind

To determine A, we Choose A such that

$$F(a) = F(a+h)$$

$$\text{or, } f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1}f^{(n-1)}(a) + hA = f(a+h) \quad (a)$$

Now, from hypothesis (i)

$f, f', f'' \text{------} f^{n-1}$ are

diff$^{=}$ in $]a, a+h[$

Also $(a+h-x), \dfrac{(a+h-x)^2}{\lfloor 2}, \text{----} \dfrac{(a+h-x)^n}{\lfloor n}$

are all diff$^{=}$ in $]a, a+h[$

Hence, from above, we can ~~say~~ so that F is also diff$^{=}$ in $]a, a+h[$

$$F'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - \frac{\cancel{2}(a+h-x)}{\cancel{\lfloor 2}} f''x + \frac{(a+h-x)^2}{\lfloor 2} f'''(x) - \frac{(a+h-x)^2}{\lfloor 2} f'''(x) + \text{-----} - \frac{(a+h-x)^{n-2}}{\lfloor n-2} f^{(n-1)}(x) + \frac{(a+h-x)^{n-1}}{\lfloor n-1} f^n(x) - \frac{(a+h-x)^{n-1}}{\lfloor n-1} A.$$

On Simplification

$$F'(x) = \frac{(a+h-x)^{n-1}}{\lfloor n-1} f^n(x) - A, \text{—} (b)$$

Hence, F satisfies all the condition of Rolle's theorem.

$$\therefore F'(a+\theta h) = 0 \text{ — } (c) \quad [0<\theta<1]$$

On putting $x = a + \theta h$ in (b), we have

$$F'(a+\theta h) = \frac{(a+h-a-\theta h)^{n-1}}{\lfloor n-1} \left[f^{(n)}(a+\theta h) - A\right] \quad \text{(d)}$$

from (c) & (d)

$$0 = \frac{(1-\theta)^{n-1} \cdot h^{n-1}}{\lfloor n-1} \left\{f^{n}(a+\theta h) - A\right.$$

$$\text{or, } A = \frac{(1-\theta)^{n-1} \cdot h^{n-1}}{\lfloor n-1} f^{n}(a+\theta h).$$

Now, putting the value of A in eqn (a), we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-1}}{\lfloor n-1} f^{n}(a+\theta h)$$

Hence $(n+1)^{th}$ term

$$= \frac{h^n}{\lfloor n} f^{n}(a+\theta h)$$

is know as Cauchy's form of remainder.